Direct decompositions of basic algebras and their idempotent modifications

Ivan Chajda
e-mail: chajda@inf.upol.cz
Palacký University Olomouc, Czech Republic

Miroslav Kolářík⋆
e-mail: kolarik@inf.upol.cz
Palacký University Olomouc, Czech Republic

1. Introduction

It is well-known that a bounded lattice $L = (L; \lor, \land, 0, 1)$ is directly decomposable into lattices $L_1, L_2$ isomorphic to the intervals $[a, 1], [b, 1]$ of $L$ if $b$ is a complement of $a$ and $a, b$ are standard elements. Since every basic algebra $A = (A; \oplus, \neg, 0)$ induces a lattice $L(A) = (A; \lor, \land)$ which is bounded by $0$ and $1 = \neg 0$, we can ask if also $A$ is directly decomposable whenever there exists a complemented and standard element of $L(A)$. In what follows we show that the condition concerning this element must be enlarged due to the fact that the operations $\oplus$ and $\neg$ cannot be derived by means of the lattice operations of $L(A)$. However, we set up a natural necessary and sufficient condition for the direct decomposability of $A$.

By a basic algebra (see e.g. [1], [2]) is meant an algebra $A = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the following four axioms

(BA1) $x \oplus 0 = x$;
(BA2) $\neg \neg x = x$;
(BA3) $\neg (\neg x \oplus y) \oplus y = \neg (\neg y \oplus x) \oplus x$;
(BA4) $\neg (\neg (x \oplus y) \oplus z) \oplus (x \oplus z) = \neg 0$.

As usual, we will write $1$ instead of $\neg 0$. We say that a basic algebra $A$ is non-trivial if $0 \neq 1$ (i.e. $|A| > 1$).

Having a basic algebra $A = (A; \oplus, \neg, 0)$, one can introduce the induced order $\leq$ on $A$ as follows

$$x \leq y \text{ if and only if } \neg x \oplus y = 1.$$ 

It is an easy exercise to verify that $\leq$ is really an order on $A$ and $0 \leq x \leq 1$ for each $x \in A$. Moreover, $(A; \leq)$ is a bounded lattice in which

$$x \lor y = \neg (\neg x \oplus y) \oplus y \text{ and } x \land y = \neg (\neg x \lor \neg y).$$

The lattice $L(A) = (A; \lor, \land)$ will be called the induced lattice of $A$. In particular for each $a \in A$ there exists an antitone involution $x \mapsto x^a$ on the interval $[a, 1]$ (called a section) where $x^a = \neg x \oplus a$.

It is well-known (see e.g. [1], [3]) that also conversely, if $(A; \lor, \land, (^a)_{a \in A}, 0, 1)$ is a bounded lattice with section antitone involutions, we are able to construct a basic algebra using the operations

$$x \oplus y = (x^0 \lor y)^0 \text{ and } \neg x = x^0.$$
Lemma 1. Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra, $\leq$ the induced order and $a \in A$. Define the polynomial operations $\neg_a$ and $\oplus_a$ on the interval $[a, 1]$ as follows
\[
\neg_a x = -x \oplus a \quad \text{and} \quad x \oplus_a y = -(\neg x \oplus a) \oplus y.
\]
Then $([a, 1]; \oplus_a, \neg_a, a)$ is a basic algebra.

The basic algebra $([a, 1]; \oplus_a, \neg_a, a)$ where the operations $\oplus_a, \neg_a$ are defined as in Lemma 1 will be called an interval basic algebra.

2. Direct decomposibility of basic algebras

Now, we will set up the conditions under which a basic algebra $\mathcal{A}$ can be directly decomposed. First, we define several concepts.

Definition 2. An element $a$ of a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ is called strong if
\[(a) \quad x \oplus a = x \lor a \quad \text{and} \quad x \oplus \neg a = x \lor \neg a\]
for every $x \in A$.

A strong element $a$ of $\mathcal{A}$ is called a decomposing element if it moreover satisfies
\[(b) \quad (x \oplus y) \oplus a = x \oplus (y \oplus a), \quad (x \oplus y) \oplus \neg a = x \oplus (y \oplus \neg a)
\]
and $x \oplus a = a \oplus x$, $x \oplus \neg a = \neg a \oplus x$
for all $x, y \in A$.

Let us note that 0 and 1 are decomposing elements for every basic algebra $\mathcal{A}$.

Recall (see [4]) that the element $a$ of a lattice $(L; \lor, \land)$ is called distributive if for all $x, y \in L$
\[
(x \land y) \lor a = (x \lor a) \land (y \lor a)
\]
and the element $a$ of a lattice $(L; \lor, \land)$ is called standard if for all $x, y \in L$
\[
x \land (a \lor y) = (x \land a) \lor (x \land y).
\]

Further, recall that if $(L; \lor, \land)$ is a lattice and $a \in L$ then the following two conditions are equivalent:
\[(a) \quad a \text{ is standard}
\]
\[(\beta) \quad a \text{ is distributive and, for } x, y \in L,
\]
\[
a \land x = a \land y \quad \text{and} \quad a \lor x = a \lor y \quad \text{imply that} \quad x = y
\]
(for more details see [4]).

Lemma 3. Let $a$ be a strong element of a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$. Then
\[(i) \quad a \text{ is boolean (i.e. } a \lor \neg a = 1, a \land \neg a = 0);\]
\[(ii) \quad a \text{ and } \neg a \text{ are distributive elements.}\]

Lemma 4. Let $a$ be a strong element of a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ and $\neg a$ be a standard element of the induced lattice $\mathcal{L}(A) = (A; \lor, \land)$. Then the mapping $
\varphi_a(x) = (x \lor a, x \lor \neg a)$ is a lattice isomorphism of $\mathcal{L}(A)$ onto the direct product of lattices $([a, 1]; \lor, \land) \times ([\neg a, 1]; \lor, \land)$. 

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Theorem 5. Let $\mathcal{A} = (A; \oplus, -, 0)$ be a basic algebra. Then $\mathcal{A}$ is isomorphic to a direct product of non-trivial basic algebras $\mathcal{B}_1, \mathcal{B}_2$ if and only if there exists a decomposing element $a \in A$, $0 \neq a \neq 1$ such that $\neg a$ is standard in the induced lattice $\mathcal{L}(A) = (A; \lor, \land)$. If it is the case then $\mathcal{A}$ is isomorphic to the direct product of interval basic algebras $([a, 1]; \oplus_a, \neg a, a)$ and $([\neg a, 1]; \oplus_{\neg a}, \neg \neg a, \neg a)$.

3. IDEMPOTENT MODIFICATION OF BASIC ALGEBRAS

The concept of idempotent modification of an algebra was introduced by J. Ježek [6] as follows.

Definition 6. An idempotent modification of an algebra $\mathcal{A} = (A; F)$ is an algebra $\mathcal{A}_f = (A; F_I)$ with the same underlying set $A$, where $|F| = |F_I|$ and for every $f \in F$ the corresponding operation $f_I \in F_I$ is defined as follows

(i) if $f$ is at most unary then $f_I = f$;
(ii) if $f$ is $n$-ary with $n > 1$ and $a_1, \ldots, a_n \in A$ then

$$f_I(a_1, \ldots, a_n) = \begin{cases} a_1 & a_1 = a_2 = \cdots = a_n \\ f(a_1, \ldots, a_n) & \text{otherwise.} \end{cases}$$

The main result of [6] is that for any group $G$ its idempotent modification $G_I$ is subdirectly irreducible.

In what follows we will treat direct decomposability of an idempotent modification of a basic algebra.

For this we slightly modify our definition of basic algebra. As mentioned above, every basic algebra $\mathcal{A} = (A; \oplus, -, 0)$ has induced lattice $\mathcal{L}(A) = (A; \lor, \land)$ where $\lor$ and $\land$ are term operations of $\mathcal{A}$. Hence, inserting $\lor$ and $\land$ into the type of $\mathcal{A}$, we obtain an algebra with the same clone of term operations and hence term equivalent to $\mathcal{A}$. From now on, by a basic algebra we will understand an algebra $\mathcal{A} = (A; \oplus, -, 0, \lor, \land)$ where the term operations $\lor$ and $\land$ are defined by $x \lor y = \neg(\neg x \oplus y) \oplus y$, $x \land y = \neg(x \lor \neg y)$.

The reason of this insertion is that when an idempotent modification of $(A; \oplus, -, 0)$ is considered, the resulting algebra does not have the lattice structure. However, if $\mathcal{A} = (A; \oplus, -, 0, \lor, \land)$ is treated then the lattice structure for $\mathcal{A}_f$ is preserved because both $\lor$ and $\land$ are idempotent operations on $A$.

Theorem 7. Let $\mathcal{A} = (A; \oplus, -, 0, \lor, \land)$ be a basic algebra whose at least one element is not boolean. Then its idempotent modification $\mathcal{A}_f = (A; \lor_I, -, 0, \lor, \land)$ is not directly decomposable.

Call a basic algebra $\mathcal{A} = (A; \oplus, -, 0)$ distributive if the induced lattice $\mathcal{L}(A) = (A; \lor, \land)$ is distributive. For example, if $\mathcal{A}$ is commutative then $\mathcal{A}$ is distributive (but not vice versa) see e.g. [1]. For distributive basic algebras, we can modify our result as follows

Corollary 8. Let $\mathcal{A} = (A; \oplus, -, 0, \lor, \land)$ be a distributive basic algebra with $|A| > 2$. Then its idempotent modification is directly indecomposable if and only if $\mathcal{A}$ contains an element which is not boolean.
REFERENCES