TIME SPENT BELOW A RANDOM THRESHOLD BY A POISSON DRIVEN SEQUENCE OF OBSERVATIONS

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Abstract

The mean and the variance of the time $S(t)$ spent by a system below a random threshold until $t$ are obtained when the system level is modelled by the current value of a sequence of independent and identically distributed random variables appearing at the epochs of a nonhomogeneous Poisson process. In the case of the homogeneous Poisson process, the asymptotic distribution of $S(t)/t$ as $t \to \infty$ is derived.

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1. Introduction

If shocks occurring in time affect the level of an economic, financial, environmental, biological or engineering system, then the proportion of time spent by the system below a threshold is frequently of interest. If $Y_i$ denotes the system level during the period between the $(i - 1)$th and $i$th shocks and if $N(t)$ counts the number of shocks during $[0, t]$ for $t \geq 0$, then the process $Z_t = \sum_{n=1}^{\infty} Y_n 1(N(t) = n - 1)$, $t \geq 0$, keeps track of the level of the system. Given any $t > 0$, we are interested in the proportion of time during $[0, t]$ when the process $Z = (Z_t)_{t \geq 0}$ is below the system’s threshold. However, there are situations in which there is no practical way to identify the system’s threshold with certainty. Therefore, the threshold will be considered a random variable $X$ and interest will center on $S(t)/t$, where $S(t)$ denotes the total time during $[0, t]$ that the process $Z$ falls below $X$. Assuming that the process $N = (N(t))_{t \geq 0}$ is a nonhomogeneous Poisson process with $N, Y = (Y_i)_{i \geq 1}$, and $X$ independent, we obtain the mean and variance of $S(t)/t$ in Section 2. The choice of the nonhomogeneous Poisson process to model the time epochs of shocks was first proposed by Esary et al. [4]. We also prove, in Section 3, that if the shock process $N$ is homogeneous Poisson, then $S(t)/t$ converges in distribution to $G(X)$, where $G$ denotes the common distribution function of the $Y_i$.

A related but easier scheme of exceedance was proposed by Wesolowski and Ahsanullah [8]. They investigated the exact and asymptotic distributions of three statistics connected with exceeding an independent random threshold in a sequence of independent and identically distributed (i.i.d.) observations. In particular, they considered a discrete analogue of our $S(t)$. Some additional distributional properties related to the exceedance scheme of [8] have been recently studied by Bairamov and Eryilmaz [1], Bairamov and Kotz [2] and Eryilmaz [3].

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2. Mean and variance

Assuming that $Y = (Y_i)_{i=1,2,...}$ is a sequence of i.i.d. observations with common distribution function $G$, one of the statistics considered in [8] was the number $S^*(n)$ of observations in a sample of size $n$ falling below the random level $X$, where $X$ is independent of $Y$. It was proved there that the conditional distribution of $S^*(n)$ given $X$ is binomial with parameters $n$ and $G(X)$. Consequently,

$$E(S^*(n)) = n E(G(X))$$

and

$$\text{var}(S^*(n)) = n E(G(X)\tilde{G}(X)) + n^2 \text{var}(G(X)),$$

where $\tilde{G} = 1 - G$.

Here, we study similar characteristics in the case when observations arrive at the epochs of a nonhomogeneous Poisson process $N = (N(t))_{t\geq 0}$ which is independent of $(Y, X)$.

The main object of our interest is the process $Z = (Z(t))_{t\geq 0}$ which keeps track of the current $Y_j$ as follows:

$$Z(t) = \sum_{n=1}^{\infty} Y_n \mathbf{1}(N(t) = n - 1), \quad t \geq 0.$$ 

Denote by $S(t)$ the time spent by the process $Z$ below the level $X$ up to the time $t$, and by $W_i$, $i = 0, 1, \ldots$, the interarrival times of the process $N$, that is, $W_i = V_{i+1} - V_i$, where $V_i = \inf\{t \geq 0 : N(t) = i\}$ is the $i$th epoch of $N$, $i = 0, 1, 2, \ldots$. Then $S(t)$ has the form

$$S(t) = \left\{ \begin{array}{ll}
\sum_{i=0}^{N(t)-1} W_i \mathbf{1}(Y_{i+1} \leq X) + \left( t - \sum_{i=0}^{N(t)-1} W_i \right) \mathbf{1}(Y_{N(t)+1} \leq X) \mathbf{1}(N(t) \geq 1) \\
+ t \mathbf{1}(Y_{N(t)+1} \leq X) \mathbf{1}(N(t) = 0) \\
= \left( \sum_{i=0}^{N(t)-1} W_i [\mathbf{1}(Y_{i+1} \leq X) - \mathbf{1}(Y_{N(t)+1} \leq X)] \right) \mathbf{1}(N(t) \geq 1) \\
+ t \mathbf{1}(Y_{N(t)+1} \leq X). 
\end{array} \right.$$ 

(1)

Though we are unable to derive the exact distribution of $S(t)$, the first two moments are computable and similar in form to the corresponding results in [8].

**Proposition 1.** In the model defined above, for any $t \geq 0$,

$$E(S(t)) = t E(G(X)),$$

(2)

$$\text{var}(S(t)) = \chi(t) E(G(X)\tilde{G}(X)) + t^2 \text{var}(G(X)),$$

(3)

where

$$\chi(t) = 2 \int_{0<x<y<t} P(N(y) = N(x)) \, dx \, dy \leq t^2.$$ 

Before proving the above proposition, we present a result on order statistics which will be used later on in the proof.
Lemma 1. Let \( X_{1:n}, \ldots, X_{n:n} \) be order statistics from an i.i.d. sample with distribution function \( F \) having support \([0, a]\). Then

\[
E \left[ X_{1:n}^2 + \sum_{i=2}^{n} (X_{i:n} - X_{i-1:n})^2 + (a - X_{n:n})^2 \right] = 2 \int_{0<x<y<a} (F(x) + \bar{F}(y))^n \, dx \, dy. \tag{4}
\]

Proof. Observe first that, for any square integrable random variable \( X \) with distribution function \( F \) having support \([0, a]\),

\[
E(X^2) = 2 \int_{0}^{a} y \bar{F}(y) \, dy = 2 \int_{0<x<y<a} \bar{F}(y) \, dx \, dy.
\]

Consequently,

\[
E((a - X)^2) = 2 \int_{0<x<y<a} F(a - y) \, dx \, dy = 2 \int_{0<x<y<a} F(x) \, dx \, dy.
\]

We proceed by induction with respect to \( n \). For \( n = 1 \), the result has just been proved. Denote the left-hand side of (4) by \( L_n \). Then

\[
L_n = E \left[ E \left( X_{1:n}^2 + \sum_{i=2}^{n} (X_{i:n} - X_{i-1:n})^2 \mid X_{n:n} \right) + (a - X_{n:n})^2 \right].
\]

It is known (see, for instance, [6, Chapter 4]) that the conditional distribution of \( (X_{1:n}, \ldots, X_{n-1:n}) \) given \( X_{n:n} = x \) is the same as the joint distribution of order statistics from an i.i.d. sample of size \( n - 1 \) based on the distribution function

\[
G_x(u) = \begin{cases} 
  \frac{F(u)}{F(x)} & \text{for } u < x, \\
  1 & \text{for } u \geq x.
\end{cases}
\]

Then, by the induction assumption, it follows that

\[
L_n = E \left( 2 \int_{0<x<y<X_{n:n}} \left[ \frac{F_X(u,y)}{F_X(u,x)} \right]^{n-1} \, dx \, dy + (a - X_{n:n})^2 \right)
\]

\[
= 2 \int_{0<x<y<a} \left( \int_{y}^{a} \left[ G_x(u) + \bar{G}_x(y) \right]^{n-1} \, du + F_X(u,x) \right) \, dx \, dy
\]

\[
= 2 \int_{0<x<y<a} \left( n \int_{F(y)}^{1} \left[ t + \frac{F(x) - F(y)}{n} - t \right]^{n-1} \, dt + F^n(x) \right) \, dx \, dy,
\]

which immediately implies the result.
Proof of Proposition 1. By the independence properties, we have

\[
E(S(t)) = \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} E(W_i [1(Y_{i+1} \leq X) - 1(Y_{n+1} \leq X)] \mid N(t) = n) P(N(t) = n) + t \sum_{n=0}^{\infty} E(1(Y_{n+1} \leq X) \mid N(t) = n) P(N(t) = n)
\]

\[
= \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} E(W_i \mid N(t) = n) [E(1(Y_{i+1} \leq X)) - E(1(Y_{n+1} \leq X))] P(N(t) = n)
\]

\[
+ t E(G(X)).
\]

The formula (2) follows since \(E(1(Y_{i+1} \leq X)) - E(1(Y_{n+1} \leq X)) = E(G(X)) - E(G(X)) = 0\).

In order to find the variance of \(S(t)\), we first compute its second conditional moment given \(N(t) = n > 0\):

\[
E(S^2(t) \mid N(t) = n) = \sum_{i=0}^{n-1} E(W_i^2 \mid N(t) = n) E((I_{i+1} - I_{n+1})^2)
\]

\[
+ \sum_{i \neq j}^{n-1} E(W_i W_j \mid N(t) = n) E((I_{i+1} - I_{n+1})(I_{j+1} - I_{n+1}))
\]

\[
+ 2t \sum_{i=0}^{n-1} E(W_i \mid N(t) = n) E(I_{n+1}(I_{i+1} - I_{n+1})) + t^2 E(I_{n+1}),
\]

where \(I_j = 1(Y_j \leq X)\) for \(j = 1, 2, \ldots\). Observe that

\[
E((I_{i+1} - I_{n+1})^2) = 2 E(G(X) \tilde{G}(X)), \quad 0 \leq i < n,
\]

\[
E((I_{i+1} - I_{n+1})(I_{j+1} - I_{n+1})) = E(G(X) \tilde{G}(X)), \quad 0 \leq i, j < n, i \neq j.
\]

\[
E(I_{n+1}(I_{i+1} - I_{n+1})) = -E(G(X) \tilde{G}(X)), \quad 0 \leq i < n.
\]

Consequently,

\[
E(S^2(t) \mid N(t) = n)
\]

\[
= E(G(X) \tilde{G}(X)) \left[ E((t - V_n)^2 \mid N(t) = n) + E\left(\sum_{i=0}^{n-1} W_i^2 \mid N(t) = n\right)\right]
\]

\[
+ t^2 E(G^2(X))
\]

for \(n > 0\). Setting \(\sum_{i=0}^{n-1} = 0\) and recalling that \(V_0 = 0\) almost surely, we find that the above formula also holds for \(n = 0\) since, by direct computation, \(E(S^2(t) \mid N(t) = 0) = t^2 E(G(X))\). Hence, for any \(t \geq 0\),

\[
\text{var}(S(t)) = E(G(X) \tilde{G}(X)) \left[ E((t - V_{N(t)})^2) + E\left(\sum_{i=0}^{N(t)-1} W_i^2 \right)\right] + t^2 \text{var}(G(X))
\]

\[
= E(G(X) \tilde{G}(X)) \left[ V_1^2 + \sum_{i=2}^{N(t)} (V_i - V_{i-1})^2 + (t - V_{N(t)})^2 \right] + t^2 \text{var}(G(X)).
\]
Now to get (3) it suffices to compute \( \chi(t) \). In order to do that, we recall (see, for instance, [7, Theorem 12.2.1]) that the conditional distribution of the random vector \((V_1, \ldots, V_{N(t)})\) given \(N(t) = n\) is equal to the joint distribution of the order statistics of a random sample of size \(n\) from the distribution

\[
H(x) = \begin{cases} 
0, & x < 0, \\
\frac{\Lambda(x)}{\Lambda(t)}, & 0 \leq x < t, \\
1, & t \leq x,
\end{cases}
\]

where \(\Lambda\) is the mean value function of the Poisson process. Now applying Lemma 1 and changing the order of integration, we obtain that

\[
\begin{align*}
E \left( E \left[ V_1^2 + \sum_{i=2}^{N(t)} (V_i - V_{i-1})^2 + (t - V_n)^2 \bigg| N(t) \right] \right) \\
= 2 \int_0^t \int_0^t \exp(A(t)[H(x) + H(y) - 1]) \, dx \, dy \\
= 2 \int_0^t \int_0^t \exp(-[\Lambda(y) - \Lambda(x)]) \, dx \, dy.
\end{align*}
\]

**Remark 1.** Observe that \(E(S(t))\) does not depend on the parameters of the driving process \(N\). On the other hand, \(\text{var}(S(t))\) depends on the intensity \(\Lambda\) of the Poisson process and increases in \(t\) at most as a quadratic function.

**Remark 2.** If \(N\) is a homogeneous Poisson process with mean value function \(\Lambda(t) = \lambda t\), where \(\lambda\) is a positive constant, then

\[
\chi(t) = 2 \int_0^t \int_0^{t-y} e^{-\lambda w} \, dw \, dy = \frac{2}{\lambda^2} (e^{-\lambda t} - 1 + \lambda t).
\]

In case \(N\) is a nonhomogeneous Poisson process with \(\Lambda(t) = \log(1 + t)\),

\[
\chi(t) = 2 \int_0^t \int_0^{t-y} \frac{1 + y}{1 + y + w} \, dw \, dy = \frac{t^2}{2} - \log(1 + t).
\]

Note that \(\chi(t)/t^2 \to 0\) as \(t \to \infty\) when \(N\) is homogeneous, but not when \(\Lambda(t) = \log(1 + t)\).

### 3. Asymptotic distribution

It was shown in [8] that \(S^*(n)/n\) converges in distribution to \(G(X)\), \(S^*(n)/n \xrightarrow{d} G(X)\) as \(n \to \infty\). Here the formulae for \(E(S(t))\) and \(\text{var}(S(t))\) suggest that a similar result cannot be true unless \(\chi(t)/t^2\) converges to 0 as \(t \to \infty\). At the same time it is reasonable to conjecture that \(S(t)/t \xrightarrow{d} G(X)\) if \(\chi(t)/t^2 \to 0\). We are not able to answer this question in full generality; however, in the case of the homogeneous Poisson process, the answer is affirmative.

**Proposition 2.** If \(N\) is a homogeneous Poisson process, then

\[
\frac{S(t)}{t} \xrightarrow{d} G(X) \quad \text{as} \quad t \to \infty.
\]
Proof. Let \( I_j = 1(Y_j \leq X) \) for \( j = 1, 2, \ldots \) and rewrite (1) as \( S(t) = S_1(t) + S_2(t) + S_3(t) \), where

\[
S_1(t) = \mathbf{1}(N(t) \geq 1) \sum_{j=0}^{N(t)-1} W_j I_{j+1},
\]

\[
S_2(t) = (t - V_{N(t)}) I_{N(t)+1} \mathbf{1}(N(t) \geq 1)
\]

and

\[
S_3(t) = t I_{N(t)+1} \mathbf{1}(N(t) = 0).
\]

Clearly, \( S_2(t) \leq t - V_{N(t)} \). But the distribution of \( t - V_{N(t)} \) is known; see, for instance, [5]. Thus, for any \( \varepsilon > 0 \) and \( t \) sufficiently large,

\[
P\left( \frac{t - V_{N(t)}}{t} > \varepsilon \right) = e^{-\lambda \varepsilon t}.
\]

Consequently, \( S_2(t)/t \xrightarrow{p} 0 \) as \( t \to \infty \).

Further, for any \( \varepsilon > 0 \),

\[
P\left( \frac{S_3(t)}{t} > \varepsilon \right) = P(I_{N(t)+1} \mathbf{1}(N(t) = 0) > \varepsilon) \leq P(N(t) = 0) = e^{-\lambda t} \to 0 \text{ as } t \to \infty;
\]

hence \( S_3(t)/t \xrightarrow{p} 0 \).

Observe now that \( \mathbf{1}(N(t) \geq 1) \xrightarrow{p} 1 \) as \( t \to \infty \). Consequently, to prove that \( S_1(t)/t \xrightarrow{d} G(X) \), it suffices to prove that

\[
\frac{1}{t} \sum_{j=0}^{N(t)-1} W_j I_{j+1} \xrightarrow{d} G(X).
\]

This will be done in two steps.

Denoting by \( \lambda \) the intensity of the Poisson process, we will first prove that

\[
\frac{1}{t} \sum_{j=0}^{\lfloor \lambda t \rfloor - 1} W_j I_{j+1} \xrightarrow{d} G(X)
\]

where \( \lfloor . \rfloor \) is the integer-part function. Note that the sequence \( (W_j I_{j+1})_{j=0,1,\ldots} \) is conditionally i.i.d. given \( X \) and \( E(W_j I_{j+1} \mid X) = G(X)/\lambda \). Then, by the law of large numbers, for any real \( \varepsilon \),

\[
E\left[ \exp\left( \frac{i t}{t} \sum_{j=0}^{\lfloor \lambda t \rfloor - 1} W_j I_{j+1} \right) \right] = E\left[ E\left( \exp\left( \frac{i \lambda t}{\lambda t} \left( \frac{\lfloor \lambda t \rfloor - 1}{\lfloor \lambda t \rfloor} \right) \sum_{j=0}^{\lfloor \lambda t \rfloor - 1} W_j I_{j+1} \mid X \right) \right) \right] \to E(e^{\varepsilon G(X)}) \text{ as } t \to \infty.
\]
Secondly, we will show that, as $t \to \infty$,

$$\frac{1}{t} \left( \sum_{j=0}^{N(t)-1} W_{j+1} - \sum_{j=0}^{[\lambda t]} W_{j+1} \right) \to 0.$$ 

To this end, observe that

$$\frac{N(t)}{[\lambda t]} = \frac{N(t)}{\lambda t} \cdot \frac{\lambda t}{[\lambda t]} \to 1 \quad \text{as} \quad t \to \infty.$$ 

Let

$$A_\varepsilon = \{(1-\varepsilon)[\lambda t] < N(t) < (1+\varepsilon)[\lambda t]\} \quad \text{for any} \quad \varepsilon > 0$$

and

$$\sigma^2 = \text{var}(W_0 I_1) = E(G(X))^2 - E(G(X)).$$

Then, for sufficiently large $t$, $P(A_\varepsilon) > 1 - \varepsilon$. Thus,

$$P \left( \left| \sum_{j=0}^{N(t)-1} W_{j+1} - \sum_{j=0}^{[\lambda t]} W_{j+1} \right| > \varepsilon \right)$$

$$\leq P \left( \left| \sum_{j=0}^{N(t)-1} W_{j+1} - \sum_{j=0}^{[\lambda t]} W_{j+1} \right| > \varepsilon t \right) \cap A_\varepsilon + \varepsilon.$$

But

$$P \left( \left| \sum_{j=0}^{N(t)-1} W_{j+1} - \sum_{j=0}^{[\lambda t]} W_{j+1} \right| > \varepsilon t \right) \cap A_\varepsilon$$

$$= P \left( \max_{\min[N(t),[\lambda t]]} \sum_{j= \min[N(t),[\lambda t]]}^{\max[N(t),[\lambda t]]-1} W_{j+1} > \varepsilon t \right) \cap A_\varepsilon$$

$$\leq P \left( \left| \sum_{j=\min[N(t),[\lambda t]]}^{(1+\varepsilon)[\lambda t]} W_{j+1} > \varepsilon t \right| \cap A_\varepsilon$$

$$\leq P \left( \left| \sum_{j=\min[N(t),[\lambda t]]}^{2 \varepsilon \lambda t} W_{j+1} > \varepsilon t \right| \cap A_\varepsilon$$

$$\leq \frac{2 \varepsilon \lambda t \sigma^2}{\varepsilon^2 t^2} \to 0 \quad \text{as} \quad t \to \infty.$$

This completes the proof of the theorem.

**Remark 3.** Observe that, in the case of nonrandom threshold, the convergence in Proposition 2 holds in probability.

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References


